Harmonia Trigonometrica;

Mathemat. Vol. 2.

OR A

SHORT TREATISE

OF

TRIGONOMETRY:

WHEREIN

The *Harmony* between *Plane* and *Spherical* Trigonometry is clearly exhibited, and thereby all the Difficulties and Perplexities of the latter are entirely removed; ſo that both are render'd equally eaſy, their ſimilar Caſes being ſolved by Theorems *materially*, and almoſt *verbally* the ſame.

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THE

PREFACE.

THE chief Deſign of this Tract (as the Title Page expreſſes) is to ſhew the *Harmony* between *plane* and *ſpherical* Trigonometry. And the better to anſwer that Purpoſe, the correſpondent Theorems of both are diſpoſed in oppoſite Pages, facing each other; whereby the Agreement between them is rendered conſpicuous at one View. This Deſign indeed takes up but a ſmall Part of the Treatiſe, but the other Particulars here added are not, 'tis hoped, without their Uſe.

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THE ELEMENTS OF TRIGONOMETRY.

CHAP. I.

DEFINITIONS.

I. TRIGONOMETRY is the Art of *meaſuring* or *reſolving Triangles.*

II. A Triangle is ſaid to be *reſolved,* when from three given Parts (-either Sides, Angles*, or both-) a fourth (-either Side or Angle-) is found out.

III. *Trigonometry* is either *plane* or *ſpherical*; the firſt reſolves *plane* Triangles, the ſecond *ſpherical* ones.

SCHOLIUM.

Inaſmuch as for the reſolving of Triangles, it is required, that the Proportion which all the Parts of a Triangle bear to each other be known, which really is not; for the Peripheries of Circles, (and conſequently the Meaſures of *Angles* in a *plane* Triangle, and the Meaſures of *Sides* and *Angles* in a *Spherical* Triangle,) could never yet be reduced to right Lines; Mathematicians therefore, in order to remedy this Defect, have applied certain right Lines to the Circle, which bear an invariable Ratio to the Radius thereof. And theſe Lines they called -*Chords* - *Sines* - *Tangents* and *Secants.*

IV. A *Chord* is a right Line drawn from one End of an Arc to the other; thus A G is the *Chord* of the Arc A E G, and of the Arc A H G.

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* Except the three Angles of a plane Triangle; for theſe determine only the *Proportion* of the Sides, and not their *Quantity.*

COROLLARY.

The *Chord* therefore of an *Arc*, and of its *Complement* to a Circle, is the *ſame*.

A *Sine* is either *right* or *verſed*.

V. A *Right Sine*, moſt commonly called, ſimply, a *Sine*, is a Perpendicular let fall from one Extremity of an Arc, upon the Radius drawn to the other Extremity of that ſame Arc; or it is one Half of the Cord of the double Arc. Thus the *Right Sine* of the *Arc* AE is the Line AC, $=\frac{1}{2}$ AG, the Chord of AEG double of AE.

COROLLARY.

As the Line AG is the Chord of the Arc AEG, and likewiſe of AHG its *Complement* to a Circle, conſequently AC ($=\frac{1}{2}$ AG) the *Sine* of the *Arc* AE, is likewiſe the *Sine* of its *Supplement* AH. The *Sine* therefore, of an *Angle*, and of its *Supplement* to 180 Degrees, or two right Angles, is the *ſame*.

SCHOLIUM.

It appears from *Def.* 5. that every Sine (i. e. right Sine) is the half of a Chord, and that the Arc of the Sine is the half of the Arc of the ſaid Chord. Wherefore, becauſe there is the ſame Proportion between two Halves, as there is between their Wholes, and the ſaid Proportion between the Halves (as being expreſſible in leſs Numbers) is much eaſier to be calculated, than the Proportion between their Wholes; hence (ſays Dr WELLS) Sines are uſed rather than Chords in Trigonometrical Calculations, and the Proportion of Sines (not of Chords) to the Radius is ſet forth in Trigonometrical Tables.

VI. A *verſed Sine* is the Segment of the Diameter intercepted between the right Sine and the Arc. Thus CE is the *verſed Sine* of the *Arc* AE, and CH the *verſed Sine* of its *Supplement* AH.

VII. The *Tangent* of an Arc is a right Line drawn perpendicular to the Diameter, touching the Arc at one End, and produced till it meets

VIII. The *Secant*, which is a right Line drawn from the Center thro' the other End of the ſaid Arc. Thus ED is the *Tangent*, and BD the *Secant* of the *Arc* AE.

SCHOLIUM.

It is to be obſerved, that as the Sine of an Arc, and of its *Supplement*, is the ſame; ſo the *Tangent* or *Secant* of any *Arc*, and of that Arc's *Supplement*, is the *ſame* likewiſe.

IX. The *Coſine*, *Cotangent*, and *Coſecant* of an *Arc*, is the *Sine*, *Tangent*, and *Secant* of that Arc which is the *Complement* of the former Arc to a Quadrant. Thus, FA, the *Sine* of AK; KL, its *Tangent*; and BL, its *Secant*; are reſpectively the *Coſine*, *Cotangent*, and *Coſecant* of the *Arc* AE.

CHAP. II.

The PROPERTIES *or* AFFECTIONS *of* Spherical *Triangles.*

DEFINITIONS.

I. A *Spherical Triangle* is that which is comprehended under three Arcs of *great Circles* interſecting each other on the Surface of the *Sphere*.

II. The *Meaſure* of a *Spherical Angle* is an Arc of a *great Circle* intercepted between the Sides comprehending the Angle, the *Pole* of that Circle being the *angular Point*.

PROPERTIES.

I. In every Spherical Triangle each Side is *leſs* than a Semicircle.

II. Any *two Sides* taken together are *greater* than the *third*.

III. The *three Sides* taken together are *leſs* than a whole Circle.

IV. The *three Angles* taken together are *greater* than *two right Angles*, and *leſs* than *ſix*.

V. The *greater Angle* is ſubtended by the *greater Side*, and the *leſſer Angle* by the *leſſer Side*.

VI. In a right-angled Spherical Triangle, the *Angles* are of the *ſame* Affection with their *oppoſite Sides*; that is, if the Sides are *equal*, *greater* or *leſs* than Quadrants, the Angles oppoſite thereto are *equal*, *greater* or *leſs* than Right Angles. And v. v.

VII. If the two *Sides* of a right-angled Spherical Triangle be of the *ſame* Affection, (and conſequently the *Angles*) that is, if they are *both leſs*, or *both greater* than a Quadrant, the *Hypotenuſe* will be *leſs* than a Quadrant.

VIII. But if the two *Sides* (and conſequently the *Angles*) be of *different* Affection, i. e. if *one* be *leſs*, and the *other greater* than a Quadrant, the *Hypotenuſe* will be *greater* than a Quadrant.

IX. If the *Hypotenuſe* be *greater* than a Quadrant, then the *Sides* comprehending the right Angle, and alſo the *Angles oppoſite* to them, are of *different* Affection; but if *leſs* than a Quadrant, they are of the *ſame* Affection.

X. Any Spherical Triangle may be transformed into another, whoſe three *Sides* are equal to the three *Angles* of the former, or their *Supplements*, if any of them be *obtuſe*. And contrariwiſe, whoſe *Angles* are equal to the *Sides* of the former, or their *Supplements* to Semicircles, if they are *greater* than Quadrants.

COROLLARY.

Hence it follows, that, when the three *Angles* of a Spherical Triangle are given, the three *Sides* of the reciprocal Triangle will be known; for the two *leſſer Sides* will be reſpectively equal to the two *leſſer*, or *acute Angles*, and the *greateſt Side* to the *Supplement* of the *greateſt*, or *obtuſe Angle*.

CHAP. III.

IN every *right-angled plane Triangle*, any of the three Sides may be made *Radius*, and then the two other Sides will be as *Sines*, *Tangents*, or *Secants*, as appears from the following

THEOREM.

If the *Hypotenuſe* be made Radius, the other two *Sides* are the *Sines* of their *oppoſite Angles*; but if *one* of the *Sides* (i. e. the *Baſe* or *Perpendicular)* be made Radius, then the *other Side* is the *Tangent* of its *oppoſite Angle*, and the *Hypotenuſe* is the *Secant* of that *ſame Angle*.

This will appear very clearly, if the Figures in the *Synopſis* be compared with *Def.* V. VII. VIII.

SCHOLIUM.

It may be of ſome uſe, to apprize the Reader in this Place, that there is in the *Trigonometrical Canon*, or *Tables*, a Triangle exactly *ſimilar* to the Triangle propoſed to be ſolved; and upon the *Similarity* of theſe Triangles all Trigonometrical Operations are grounded. Wherefore the Triangle propoſed to be reſolved is always conſidered in a *double Reſpect*:

—1. As

—1. As the *Given* Triangle — and 2. As the *Canonical* Triangle — and then *each Side* (as it is obſervable in the forementioned Figures) has *two* Denominations; the one *fixed* and *unchangeable*, viz. *Hypotenuſe*, *Baſe* and *Perpendicular*; the other continually *changing*, according to the Side made *Radius*. But the Denomination of any Side (under this ſecond Conſideration) may be readily known by the Word annexed to it.

CHAP. IV.

The Trigonometrical Solutions of right-angled plane *Triangles.*

THE *Caſes* of *right-angled* Triangles are *ſix*; all which may be ſolved by the following *Theorems.*

THEOREM I.

As Radius,
Is to the Hypotenuſe;
So is the Sine of either of the oblique Angles,
To its oppoſite Side.
viz. R : AC :: S.C : AB. Fig. 1.

THEOREM II.

As Radius,
Is to the Baſe;
So is the Tangent of the adjacent Angle,
To the Perpendicular.
viz. R : BC :: T.C : AB. Fig. 2.

SCHOLIUM.

By introducing *Secants* into the Analogy, we have this additional *Theorem* for the Solution of all right-angled Triangles, *viz.*

As the Baſe,
Is to Radius;
So is the Hypotenuſe,
To the Secant of the adjacent Angle. *See Fig.* 2, 3.

CHAP.

CHAP. IV.

The Trigonometrical Solutions of right-angled Spherical *Triangles.*

THE *Caſes* of *right-angled* Triangles are *ten*; all which may be ſolved by the two following *Theorems.*

THEOREM. I.

As Radius,
Is to the Sine of the Hypotenuſe;
So is the Sine of either of the oblique Angles,
To the Sine of its oppoſite Side.
viz. R : S.AC : : S.C : S.AB. Fig. 5.

COROLLARY.

Hence it follows, that, if

1. An *Angle* and its *oppoſite Side*	be given,	The *Hypotenuſe*	may be found.
2. An *Angle* and the *Hypot.*		The *oppoſite Side*	
3. The *Hypot.* and one *Side*		The *oppoſite Angle*	

THEOREM II.

As Radius,
Is to the Sine of the Baſe;
So is the Tangent of the adjacent Angle,
To the Tangent of the Perpendicular.
viz. R : S.BC : : T.C : T.AB. Fig. 5.

COROLLARY.

Hence it follows, that, if

1. The *two Sides*	be given,	The *two acute Angles*	may be found.
2. An *Angle* and the *adjacent Side*		The *other Side*	

SCHOLIUM.

Tho' there is an exact *Harmony* between *plane* and *ſpherical* Trigonometry, with regard to the *Theorems* whereby this *Species* of Triangles is reſolved; yet it muſt ſtill be confeſſed, that, with regard to the *Solution*, there is this *ſmall Difference* between them; *viz.* that in reſolving *plane Triangles* we may uſe either *Sines* or *Tangents* (i. e. Theorem 1. or 2.) indifferently; whereas in *ſpherical Trigonometry*, the Caſe that is ſolvable by *Sines* will not admit of a Solution by *Tangents*, nor that which is ſolvable by *Tangents* of a Solution by *Sines*.

CHAP.

CHAP. V.

Of the Solution *of* right-angled *Triangles, by the five Circular Parts.*

DEFINITIONS.

I. EVERY Triangle conſiſts of *ſix* Parts, viz. *three Sides* and *three Angles*; but omitting the *right Angle*, as being always known, the *five* remaining Parts, *viz.* the *Hypotenuſe*, the *two oblique Angles*, and the *Complements* of the *two Sides*, are called *Circular Parts.*

II. In the Reſolution of every Triangle, *three* of theſe Parts come always under Conſideration, *viz.* the *two* Parts *given*, and the *third required*; that which is ſituated in the *Middle*, between the other two, is called the *Middle Part*; and the other two, between which it is ſituated, are called *Extreme Parts.*

III. When the *Extreme Parts* lie *contiguous*, or are *conjoined* to the *Middle Part*, they are called *Extremes Conjunct:*

IV. But when they are *disjoined* from it, that is, when *another Part* interpoſeth on *both Sides* between the *Extreme Parts* and the *Middle Part*, then they are called *Extremes Disjunct.*

COROLLARY.

Therefore, if

1. ɔAB	be the *middle Part*, then will	1. ɔBC, A	be *Extremes conjunct*, and	1. AC, C	*Extremes disjunct.*
2. A		2. AC, ɔAB		2. ɔBC, C	
3. AC		3. A, C		3. ɔAB, ɔBC	
4. C		4. AC, ɔBC		4. ɔAB, A	
5. ɔBC		5. C, ɔAB		5. AC, A	

NB. ɔAB, &c. ſignifies the *Complement* of AB, &c.

SCHOLIUM.

The Right Angle being always known, is therefore ejected out of the Circular Parts, and the Sides by which it is comprehended are conſidered as immediately adjoining: and hence it comes to paſs, that tho' the right Angle ſtands between BC (Caſe 1.) and AB (Caſe 5.) and the middle Part, yet they are ſuppoſed to be really conjunct. Theſe Things premiſed, all the Caſes of right-angled Triangles may be eaſily and expeditiouſly ſolved by the following *Theorems.*

THEO-

THEOREM I.

In all right-angled Spherical Triangles, the Rectangle under the *Radius*, and the *Coſine* of the *middle Part*, is equal to the Rectangle under the *Sines* of the *Extremes disjunct*.

COROLLARY.

Hence it is evident, that, if from the *Sum* of the Logarithms of the *Radius* and the *Coſine* of the *middle Part*, you *ſubtract* the Logarithm of the *Sine* of *either* of the *Extremes disjunct*, the *Remainder* will be the Logarithm of the *Sine* of the *other*.

THEOREM II.

The Rectangle under the *Radius*, and the *Coſine* of the *middle Part*, is equal to the Rectangle under the *Cotangents* of the *Extremes conjunct*.

COROLLARY.

Hence it follows, that, if from the *Sum* of the Logarithms of the *Radius* and the *Coſine* of the *middle Part*, you *ſubtract* the Logarithm of the *Cotangent* of *either* of the *Extremes conjunct*, the *Remainder* will be the Logarithm of the *Cotangent* of the *other*.

SCHOLIUM I.

When a Complement happens to *concur* with a Complement in the circular Parts, then the *Sine*, or the *Tangent*, inſtead of the *Coſine* or *Cotangent*, is always to be uſed in the Proportion; for the *Coſine* or *Cotangent* of the *Complement* of an Arc, is the *Sine* or *Tangent* of *that Arc*.

SCHOLIUM II.

If we conſider the *Sides* of *plane* Triangles as the *Sines* or *Tangents* of the *Sides* of *ſpherical* ones, and apply to *thoſe* what has been here advanced of *theſe*, the Harmony between both will be ſtill conſpicuous. *See* Wolfius's Trigonometry.

SCHOLIUM III.

The Lord NAPIER, who was the firſt Inventor of this Method, makes uſe of the *Sides themſelves*, and for the *Hypotenuſe* and *oblique Angles* inſerts their *Complements* in the circular Parts; whence he deduces theſe *Theorems*.

CHAP. VI.

The Trigonometrical Solutions *of* oblique-angled plane *Triangles.*

THE *Caſes* of *oblique-angled* Triangles are *four*, all which may be ſolved by the following *Theorems.*

THEOREM I.

As one Side,
Is to the other;
So is the Sine of the Angle oppoſite to the firſt Side,
To the Sine of the Angle oppoſite to the other.
viz. AB : AC : : S.C : S.B. Fig. 4.

COROLLARY.

Hence it appears that, if

1. Two *Sides* and the *Angle oppoſ.* to one of them	be given,	1. The *Angle oppoſite* to the other	may be found.
2. Two *Angles* and the *Side oppoſ.* to one of them		2. The *Side oppoſite* to the other	

LEM-

THEOREM I.

The Rectangle under the *Radius*, and the *Sine* of the *middle Part*, is equal to the Rectangle under the *Coſines* of the *Extremes diſjunct.*

THEOREM II.

The Rectangle under the *Radius* and the *Sine* of the *middle Part*, is equal to the Rectangle under the *Tangents* of the *Extremes conjunct.*

CHAP. VI.

The Trigonometrical Solutions *of* oblique-angled Spherical *Triangles.*

THE *Caſes* of *oblique-angled* Triangles are *ſix*; all which may be ſolved by the following *Theorems.*

THEOREM I.

As the Sine of one Side,
Is to the Sine of the other;
So is the Sine of the Angle oppoſite to the firſt Side,
To the Sine of the Angle oppoſite to the other.
viz. S.AB : S.AC :: S.C : S.B. Fig. 6.

COROLLARY.

Hence it appears that if

1. Two *Sides* and the *Angle oppoſ.* to one of them
2. Two *Angles* and the *Side oppoſ.* to one of them

be given,

1. The *Angle oppoſite* to the other
2. The *Side oppoſite* to the other

may be found.

LEMMA.

If any one Angle of a plane Triangle be ſubtracted from 180 Degrees or 2 Right Angles, the Remainder will be the *Sum* of the other two Angles, (32. El. 1. Cor.) which divided by 2 gives their *Half Sum* in the Quotient.

THEOREM II.

As Half the Sum of two Sides,
Is to Half their Difference;
So is the Tangent of Half the Sum of their oppoſite Angles,
To the Tangent of Half their Difference.

$$\text{viz. } \frac{BC+AB}{2} : \frac{BC-AB}{2} :: T.\frac{A+C}{2} : T.\frac{A-C}{2}$$

Fig. 4.

COROLLARY.

From hence it follows, that, if *two Sides* and the *Angle included between them* be given, each of the other two *Angles* may be found; for their *Half Sum* will be found by *Lemma*, and their *Half Difference* by *Theorem* 2. from whence (by *Problem* p. 20.) the Angles *themſelves* may be found.

SCHOLIUM. I.

Inſtead of *Theorem* II. the following may be uſed, viz.

1. As the leſſer Side,
Is to the greater;
So is Radius,
To the Tangent of an Arc.
2. As Radius,
Is to the Tangent of that Arc, leſs 45°;
So is the Tangent of Half the Sum of their oppoſite Angles,
To the Tangent of Half their Difference.

LEMMA.

As the Coſine of Half the Sum of two Sides,
Is to the Coſine of Half their Difference;
So is the Cotangent of Half the included Angle,
To the Tangent of Half the Sum of their oppoſite Angles.

viz. Coſ. $\frac{BC+AB}{2}$: Coſ. $\frac{BC-AB}{2}$: : Cot. $\frac{B}{2}$: T. $\frac{A+C}{2}$

Fig. 6.

THEOREM II.

As the Tangent of Half the Sum of two Sides,
Is to the Tangent of Half their Difference;
So is the Tangent of Half the Sum of their oppoſite Angles,
To the Tangent of Half their Difference.*

viz. T. $\frac{BC+AB}{2}$: T. $\frac{BC-AB}{2}$: : T. $\frac{A+C}{2}$: T. $\frac{A-C}{2}$

Fig. 6.

* i. e. when the Sum of the two Sides is *leſs* than a *Semicircle*; when *greater*, take their *Supplements*, and the Operation will produce the *Supplements* of the Angles ſought to two right Angles.

COROLLARY.

Hence it follows, that, if *two Sides* and the *included Angle* be given, each of the other two *Angles* may be known, for their *Half Sum* will be found by the *Lemma*, and their *Half Difference* by the *Theorem*, from whence (by *Problem* p. 20.) the Angles *themſelves* will be known.

SCHOLIUM.

Inſtead of *Theorem* II. the following may be uſed, viz.

1. As the Sine of the leſſer Side,
Is to the Sine of the greater;
So is Radius,
To the Tangent of an Arc.
2. As Radius,
Is to the Tangent of that Arc, leſs 45°;
So is the Tangent of Half the Sum of their oppoſite Angles,
To the Tangent of Half their Difference.

From

SCHOLIUM II.

This Theorem is purpoſely inſerted on account of its great Uſe i Aſtronomy.

From the greateſt Angle of an oblique Triangle let fall a Perpendicular on the Baſe, dividing it into two Segments, then

THEOREM III.

As Half the Baſe
Is to Half the Sum of the other two Sides;
So is Half the Difference of thoſe Sides,
To Half the Difference of the Segments of the Baſe.

viz. $\frac{BC}{2} : \frac{AB+AC}{2} :: \frac{AB-AC}{2} : \frac{BD-DC}{2}$ Fig. 4

COROLLARY.

From hence it follows, that, if the *three Sides* be given, the *thre* *Angles* may be found. For the *Segments* of the Baſe will be given by this *Theorem:* and now the oblique Triangle being reduced into two right-angled ones, wherein the *Baſe* and *Hypotenuſe* are known, the *Angle* will thence be given by the Reſolution of *Caſe* V. of *right-angle* Triangles.

SCHOLIUM.

The Fractions which commonly happen in the Segments may cauſ an Error of ſome Minutes in the Angles; wherefore the following *Theorem* is to be preferred (both for *Accuracy* and *Expedition)* in the Solution of this Caſe.

THEOREM.

As the Rectangle under the Sides comprehending the Angl ſought,
Is to the Square of the Radius;

From the greateſt Angle of an oblique Triangle, let fall a Perpendicular on the Baſe, which will divide it into two Segments; then,

THEOREM III.

As the Tangent of Half the Baſe,
Is to the Tangent of Half the Sum of the other two Sides;
So is the Tangent of Half the Difference of thoſe Sides,
To the Tangent of Half the Difference of the Segments of the Baſe.

viz. $T\frac{BC}{2} : T\frac{AB+AC}{2} :: T\frac{AB-AC}{2} : T\frac{BD-DC}{2}$. Fig. 6.

COROLLARY.

From hence it appears, that, if the *three Sides* be given, either of the *three Angles* may be found. For the *Segments* of the Baſe will be given by this *Theorem*; wherefore the oblique Triangle being reduced into two right-angled ones, wherein the *Baſe* and *Hypotenuſe* are known, the *Angles* will be given by the Reſolution of *Caſe* III. of *right-angled* Triangles.

SCHOLIUM.

As the Reſolution of this Caſe by the foregoing Method is very tedious, we ſhall therefore add the following *Theorem*; by which it may be ſolved at one Operation.

THEOREM.

As the Rectangle under the Sines of the Sides comprehending the Angle ſought,
Is to the Square of the Radius;

So

So is the Rectangle under the Differences between thoſe Sides and th
Half Sum of the three Sides,

To the Square of the Sine of Half the Angle ſought.

viz. $AB \times AC : Rq :: X \times Z : Sq. \frac{A}{2}$ Fig. 4.

$$N.B.\quad \frac{BC+AB+AC}{2}\left\{\begin{array}{l}-AC=X.\\ -AB=Z.\end{array}\right.$$

P R O B L E M.

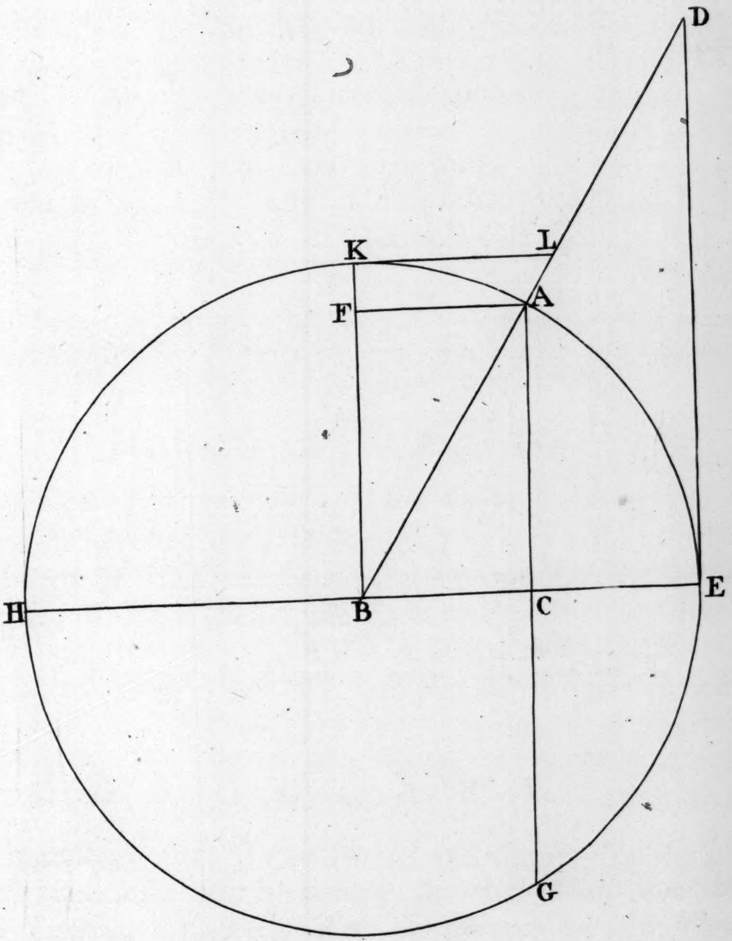
The Half Sum and Half Difference of any two Quantities bein
given, to find the Quantities themſelves.

S O L U T I O N.

1. To Half the Sum *add* Half the Difference, the Aggregate (Sur
of both) will be the *greater* Quantity.

2. From Half the Sum *ſubtract* Half the Difference, the Remainde
will be the *leſſer* Quantity.

CHAP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G R III

So is the Rectangle under the Sines of the Differences between thoſe Sides and the Half Sum of the three Sides,
To the Square of the Sine of Half the Angle ſought.

viz. $S.AB \times S.AC : Rq :: S.X \times S.Z : Sq. \frac{A}{2}$ Fig. 5.

N.B. $\left.\frac{BC+AB+AC}{2}\right\} \begin{array}{l} -AC=X. \\ -AB=Z. \end{array}$

L E M M A.

As the Coſine of Half the Sum of two Angles,
Is to the Coſine of Half their Difference;
So is the Tangent of Half the interjacent Side,
To the Tangent of Half the Sum of the other two Sides.*

viz. $Coſ. \frac{C+B}{2} : Coſ. \frac{C-B}{2} :: T. \frac{BC}{2} : T. \frac{AB+AC}{2}$ Fig. 6.

* i. e. When the SUM of the Angles is *leſs* than two *right Angles* — when *greater*, take their *Supplements*, and the Operation will produce each Side's *Supplement* to a Semicircle.

T H E O R E M IV.

As the Tangent of Half the Sum of two Angles,
Is to the Tangent of Half their Difference;
So is the Tangent of Half the Sum of their oppoſite Sides,
To the Tangent of Half their Difference.

viz. $T. \frac{C+B}{2} : T. \frac{C-B}{2} :: T. \frac{AB+AC}{2} : T. \frac{AB-AC}{2}$ Fig. 6.

C O R O L L A R Y.

From hence it follows, that if *two Angles* and the *interjacent Side* be given, the other two *Sides* may be found. for their *Half Sum* will be known by the *Lemma*, and their *Half Difference* by the *Theorem*; whence (by *Problem* p. 20) the Sides *themſelves* will be known.

C H A P. VII.

Of the Solutions *of* Trigonometrical Problems.

IN all *Trigonometrical Operations* there are always *three* Things given, to find a *fourth*; (ſuppoſing the *Radius* in *right-angled* Triangles to be one). Theſe three given Terms are to be ſo ſtated, that the ſame Proportion there is between the firſt and ſecond, may likewiſe exiſt between the third and fourth, which is always the Term *ſought*; and therefore anciently (as in all other Caſes in the Rule of Proportion) the Method was, to multiply the ſecond and third Terms into each other, and divide the Product by the firſt; the Quotient was the Anſwer to the Queſtion. But now ſince the admirable Invention of *Logarithms* (the Nature of which is ſuch that *Addition* performs the Office of *Multiplication*, and *Subtraction* of *Diviſion)* there needs only to *add* the *Logarithms* of the ſecond and third Terms together, and from their Sum to *ſubtract* the *Logarithm* of the firſt Term, and the Remainder will be the *Logarithm* of the fourth Term required.

But for more Eaſe and Expedition,

When the Radius is not in the Proportion, then, inſtead of the Logarithm of the firſt Term, *itſelf*, ſet down its *Arithmetical Complement* (i. e. what each Figure but the laſt wants of 9, and that of 10.) and *add* all the *three* Terms together, the Sum, (ſubtracting Radius,) is the *Logarithm* of the fourth Term ſought.

If the Logarithm of the Term ſought ſhould not preciſely correſpond with any Logarithm in the Tables, it is ſufficient for ordinary Purpoſes to take the neareſt to it. But where great Exactneſs is required, you muſt proportion the Difference as the Nature of the Caſe demands.

In ſtating the Terms of the Analogies, you are carefully to obſerve the following

P R E C E P T S.

I. When a *Side* is required, begin with an *Angle*; and, on the contrary, when an *Angle* is required, begin with a *Side*; then,

II. Compare a *Side* to its *oppoſite Angle*, and an *Angle* to its *oppoſite Side*.

SCHOLIUM.

By an Angle I mean the *Sine*, *Tangent*, or *Secant* of it. By a Side, (in Sph. Trigon.) the *Sine* or *Tangent* of that Side.

PROBLEM I.

In a *right-angled plane* Triangle, any *two Parts* (as the *Side* BC, and the *Angle* A) being given, to find the *other Parts*. V. *Synopſis*, *p.* 28.

SOLUTION.

I. Conſider which Side is moſt proper to be made Radius, whether the *Hypotenuſe*, *Baſe*, or *Perpendicular*.

II. If the *Hypotenuſe* be made Radius, (Fig. 1.) the Analogy will be

1. S.A : BC : : R : AC.
2. S.A : BC : : S.C : AB.

III. If the *Baſe* be made Radius, (Fig. 2.) the Proportion is

1. R : BC : : T.C : AB.
2. R : BC : : Sec.C : AC.

IV. If you make the *Perpendicular* Radius, (Fig. 3.) the Solution is performed by the following Analogy, viz.

1. T.A : BC : : R : AB.
2. T.A : BC : : Sec.A : AC.

PROBLEM II.

In a *right-angled Spherical* Triangle, any *two Parts* (beſides the right Angle) being given to find the *other Parts*. V. *Synopſis*, *p.* 30.

The SOLUTION *by the* common Rules.

I. Firſt conſider whether the *Parts* concerned in the Queſtion be *Extremes conjunct* or *disjunct*.

II. If the *Parts* be *Extremes disjunct*, and *oppoſite* to each other;

e. g. if the *Hypotenuſe* AC and the *Angle* C be given, to find its *oppoſite Side* AB; then ſay by *Theorem* I.

As Radius,
Is to the Sine of the Hypotenuſe AC;
So is the Sine of the Angle C,
To the Sine of its oppoſite Side AB.

III. But if the *Parts* be *Extremes disjunct*, and *not* oppoſite; for Example, if the *Side* BC, and the *adjacent Angle* C, be given, to find the *other Angle* A: Then the Sides of the Triangle are to be continued till they become Quadrants, and thereby form a *new* Triangle, wherein the *Parts* concerned in the Problem are *Extremes disjunct*, and *oppoſite*, as the Triangle FCG; in which are given GC, the *Complement* of the Side BC, and the *Angle* C, to find GF, the *Complement* of the Angle A; therefore ſay as before,

As Radius,
Is to the Sine of the Hypotenuſe GC. i. e. the Coſ. of BC;
So is the Sine of the Angle C,
To the Sine of its oppoſite Side FG. i. e. the Coſ. of A.

IV. If the *Parts* be *Extremes conjunct*, and the *Hypotenuſe out* of the Queſtion; for inſtance, if the *Sides* AB and BC, be given, to find the *Angle* C, the Rule is,

As the Sine of BC,
Is to the Radius;
So is the Tangent of AB,
To the Tangent of C. *Theorem 2d.*

V. But if the *Parts* be *Extremes conjunct*, and the *Hypotenuſe* concerned *in* the Problem; as if the *Hypotenuſe* AC, and the *Angle* C, were given, to find the *adjacent Side* BC: Produce each Side of the Triangle to a Quadrant, in order to find a *new* Triangle, wherein the *Hypotenuſe* is *excluded out of* the Problem, as the Triangle EAK, wherein are given EA, the *Complement* of the Hypotenuſe AC; EK, the *Complement* of the Angle C; to find the *Angle* K, the *Complement* of the Side ſought BC; therefore the Rule is the ſame as before,

As the Sine of EK, viz. Coſ. of C,
Is to the Radius;
So is the Tangent of EA, viz. Cot. of AC,
To the Tangent of K, viz. Cot. of BC.

VI. When the Sides of a Triangle are to be produced, it matters not towards which Part you produce them, if *neither* of the *acute Angles* are concerned in the Queſtion. When *one* is concerned, they muſt always be produced thro' the *other* Angle; but when *both* the Angles enter

ter the Problem, the Sides are to be produced thro' *that* Angle which is *adjacent* to the Side in Queſtion.

The SOLUTION *by the* Catholic Propoſition.

I. Firſt conſider, as before, whether the *Parts* in Queſtion be *Extremes conjunct* or *disjunct*.

II. If *one* or *both* the *Sides* comprehending the right Angle are concerned in the Problem, then, inſtead of the Sides *themſelves*, ſet down their *Complements* to a Quadrant; wherefore,

III. Since, by the *Catholic Propoſition*, the *Rectangle* under the *Radius* and the *Coſine* of the *Middle Part*, is equal to the *Rectangle* under the *Sines* of the *Extremes disjunct* — and the *Cotangents* of the *Extremes conjunct:* If from the *Sum* of the Logarithms of *two Parts* you *ſubtract* the Logarithm of the *third*, the *Remainder* will be the Logarithm of the *Sine* or *Tangent* of the *Side* or *Angle* ſought. *See Corol. to Th.* I. and II. p. 13.

PROBLEM III.

In the Reſolution of *right-angled* Spherical Triangles, to determine the *Species* of the *Side* or *Angle* found.

SOLUTION.

1. When the *Hypotenuſe* and *either* of the *Angles* are given, the Species of the *other Parts* may be determined by *Properties* IX and VI. Therefore *Caſe* I. and II. are free of all Ambiguity.

2. If the *Hypotenuſe* and *either Side* be given, as in *Caſe* III. and IV. the Affection of the *other Parts* is determined, as before, by *Properties* IX. and VI.

3. When *both* the *Sides*, or *both* the *Angles* are given, the Species of the *other Parts* are known by *Properties* VI. VII. VIII. Hence *Caſes* IX. and X. are clear.

4. If a *Side* (*Leg*) and an *Angle adjacent* thereto be given, as in *Caſe* V. and VII. then the Species of the *oppoſite Side* may be determined by *Property* VI. and from thence the Species of the *Hypotenuſe* by *Properties* VII. and VIII.

5. But

5. But when a *Side* and an *Angle oppoſite* thereto are given, then th Species of the *unknown Parts* cannot be determined. Therefore *Caſe* VI. and VIII. are wholly *ambiguous.*

PROBLEM IV.

In the Solution of *oblique-angled* Spherical Triangles, to determin the *Species* of the *Side* or *Angle* found.

SOLUTION.

1. When two *Angles* and a *Side oppoſite* to *one* of them ar given, to find the *Side oppoſite* to the *other*, as in *Caſe* I. proceed thus: Add both the given *Angles* together, and obſerve thei Sum; then to the given *Side* add the *leſſer* Arc of the *Side* found, a alſo its *greater* Arc, or *Complement*; and *that* Arc of the *Side found* which, together with the *Side given*, is of the *ſame* Affection with th Sum of the *Angles* (i. e. whoſe Sum is either *greater* or *leſs* than a Semicircle, as the Sum of the Angles is *greater* or *leſs* than two right-Angles) is the *true* Arc required.

But if the Sum of the Side given and *either* Arc of the Side foun be of the *ſame* Species with the Sum of the Angles, then the Caſe i *ambiguous.*

2. In like manner may the Ambiguity in *Caſe* IV. (where two *Sides* and an *Angle oppoſite* to *one* of them are given, to find the *Angle oppoſite* to the *other*) be cleared and determined — for *that* Value (whether the *greater* or *leſſer*) of the *Angle found*, which, together with the *Angle given*, is of the *ſame* Affection with the Sum of the *Sides* (i. e. whoſe Sum is *greater* or *leſs* than two right Angles, as the Sum of the *Sides* is *greater* or *leſs* than a Semicircle) is the *true* Value required.

But if the Sum of the Angle given and *either* Value of the Angle found, be of the *ſame* Species with the Sum of the Sides, then the Caſe is *ambiguous.*

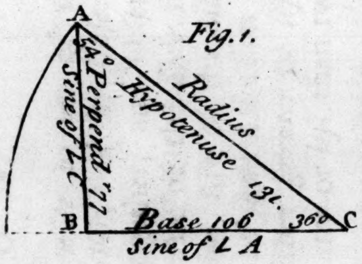
3. The other Caſes, when ſolved by the Method here propoſed, are clear of all Ambiguities.

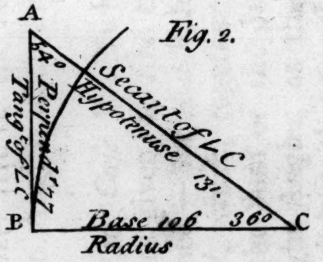
SCHOLIUM.

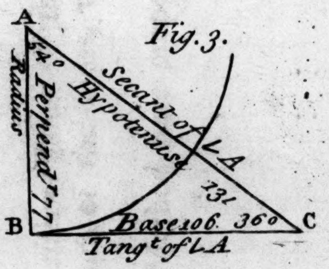
But it may not be improper to remark, that a Caſe which is ambiguous in *Trigonometry*, is very often not ſo, when it occurs in *Geography* or *Aſtronomy*, &c. for the peculiar Nature of the Subject to which it is applied, may ſometimes determine it, when the Principles of Trigonometry cannot.

A Sy-

A SYNOPSIS *of the Doctrine of* right-angled *plane Triangles.*







Caſes	Given	Sought	The Hypotenuſe Radius.	The Baſe Radius.	The Perpendicular Radius.
I	B C	A B	S.A : BC :: S.C : AB	R : BC :: T.C : AB	T.A : BC :: R : AB
	A.C	A C	S.A : BC :: R : AC	R : BC :: Sec.C : AC	T.A : BC :: Sec.A : AC
II	A B	B C	S.C : AB :: S.A : BC	T.C : AB :: R : BC	R : AB :: T.A : BC
	A.C	A C	S.C : AB :: R : AC	T.C : AB :; Sec. C : AC	R : AB :: Sec.A : AC
III	A C	B C	R : AC :: S.A : BC	Sec.C : AC :: R : BC	Sec.A : AC :: T.A : BC
	A.C	A B	R : AC :: S.C : AB	Sec.C : AC :: T.C : AB	Sec.A : AC :: R : AB
IV	B C	A.C		BC : R :: AB : T.C then	AB : R :: BC : T.A then
	A B	A C		R : BC :: Sec.C : AC	R : AB :: Sec.A : AC
V	A C	A.C	AC : R :: BC : S.A then	BC : R :: AC : Sec.C then	
	B C	A B	R : AC :: S.C : AB	R : BC : T.C : AB	
VI	A C	A.C	AC : R :: AB : S.C then		AB : R :: AC : Sec.A then
	A B	B C	R : AC :: S.A : BC		R : AB :: T.A ; BC

A SY-

SYNOPSIS *of the Doctrine of* oblique-angled *plane Triangles.*

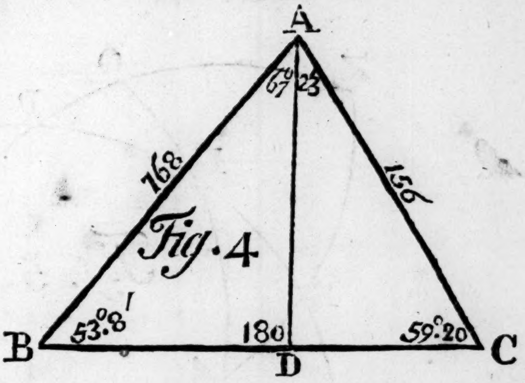


Fig. 4

	Given	Sought	
I	A.B.C BC	A B A C	S.A : BC : : S.C : AB S.A : BC : : S.B : AC
I	AB.AC B	A.C BC	AC : S.B : : AB : S.C. then 180—B—C=A. then S.B : AC : : S.A : BC.
II	AB.AC A	B.C BC	$\frac{AB+AC}{2} : \frac{AB-AC}{2} : T.\frac{C+B}{2} : T.\frac{C-B.}{2}$ then S.B : AC : : S.A : BC.
V	AB BC AC	A B C	$\frac{BC}{2} : \frac{AB+AC}{2} : : \frac{AB-AC}{2} : \frac{BD-DC}{2}$ hence BD , DC are found, and the Angles by Caſe V. of right-angled Triangles. Otherwiſe thus. $AB\times AC : Rq : : X\times Z : Sq\,\frac{A}{2}$ and ſo for any other. *N.B.* $\frac{BC+AB+AC}{2}\begin{cases}-AC=X\\ -AB=Z\end{cases}$

A SYNOPSIS *of the Doctrine of* right-angled *Spherical Tria*

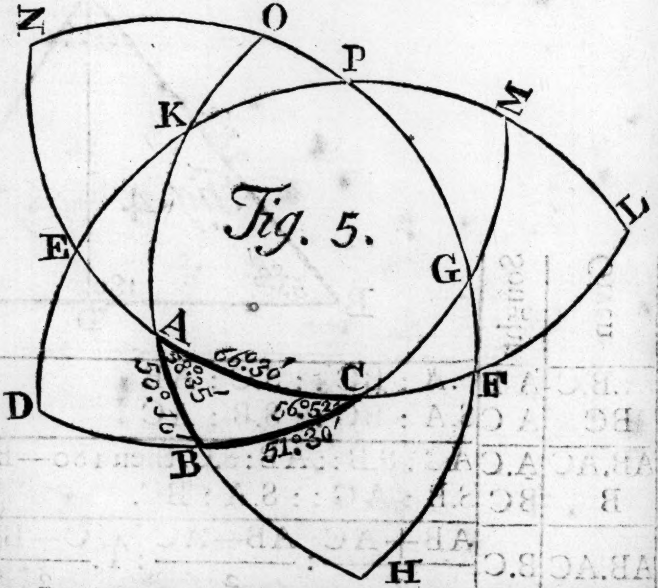


Fig. 5.

Cases	Parts given	Parts sought	Quadrantal Triangles		
I	A C	B C	A H F	S.A F : S.A C :: S.F H : S.B C	R : S.A C :: S.A : S.B C
	A	A B	G H B	S.G F : S.G H :: T.C F : T B H	Coſ.A : R :: Cot.A C : Cot. A B
		C	C L M	S.C F : S.C L :: T.G F : T.L M	Coſ.A C : R :: Cot.A : T.C
II	A C	A B	E D C	S.E C : S.A C :: S.E D : S.A B	R : S.A C :: S.C : S.A B
	C	B C	K D B	S.K E : S.K D :: T.E A : T.D B	Coſ.C : R :: Cot.A C : Cot.B C
		A	C L M	S.C L : S.C F :: T.L M : T.F G	R : Coſ.A C :: T.C : Cot.A
III	A C	A B	G H B	S.G C : S.G B :: S.C F : S.B H	Coſ.B C : R :: Coſ.A C : Coſ.A B
	B C	A	A H F	S.A C : S.A F :: S.B C : S.H F	S.A C : R :: S.B C : S.A
		C	K D B	T.D B : T.E A :: S.K D : S. K E	Cot.B C : Cot.A C :: R : Coſ.C
IV	A C	B C	K D B	S.K A : S.K B :: S.A E : S.B D	Coſ.A B : R :: Coſ.A C : Coſ.B C
	A B	C	C D E	S.A C : S.E C :: S.A B : S.E D	S.A C : R :: S.A B : S.C
		A	G H B	T.B H : T.C F :: S.G H : S.G F	Cot.A B : Cot.A C :: R : Coſ.A
V	B C	A C	K D B	S.K D : S.K E :: T.D B : T.E A	R : Coſ.C :: Cot.B C : Cot.A C
	C	A B	C D E	S.C D : S.C B :: T.E D : T.A B	R : S.B C :: T.C : T.A B
		A	C L M	S.C M : S.C G :: S.M L : S.G F	R : Coſ.B C :: S.C : Coſ.A
VI	B C	A C	A H F	S.H F : S.B C :: S.A F : S.A C	S.A : S.B C :: R : S.A C
	A	A B	A H F	T.H F : T.B C :: S.A H : S.A B	T.A : T.B C :: R : S.A B
		C	C L M	S.C G : S.C M :: S.G F : S.M L	Coſ.B C : R :: Coſ.A : S.C
VII	A B	A C	G H B	S.G H : S.G F :: T.H B : T.F C	R : Coſ.A :: Cot.A B : Cot.A C
	A	B C	A H F	S.A H : S.A B :: T.H F : T.B C	R : S.A B :: T.A : T.A C
		C	A N O	S.A O : S.A K :: S.O N : S.K E	R : Coſ.A B :: S.A : Coſ.C
VIII	A B	A C	C D E	S.E D : S.A B :: S.E C : S.A C	S.C : S.A B :: R : S.A C
	C	B C	C D E	T.E D : T.A B :: S.D C : S.B C	T.C : T.A B :: R : S.B C
		A	A N O	S.A K : S.A O :: S.K E : S.O N	Coſ.A B : R :: Coſ.C : S.A
IX	A B	A C	K D B	S.K B : S.K A :: S.B D : S.A E	R : Coſ.A B :: Coſ.B C : Coſ.A C
	B C	A	A H F	S.A B : S.A H :: T.B C : T.H F	S.A B : R :: T.B C : T.A
		C	C D E	S.B C : S.D C :: T.A B : T.D E	S.B C : R :: T.A B : T.C
X	A	A C	L M	T.L M : T.F G :: S.C L : S.C F	T.C : Cot.A :: R : Coſ.A C
	C	B C	C L M	S.L M : S.F G :: S.C M : S.C G	S.C : Coſ.A :: R : Coſ.B C
		A B	A N O	S.O N : S.K E :: S.A O : S.A K	S.A : Coſ.G :: R : Coſ.A B

SYNOPSIS *of the Doctrine of oblique-angled Spherical Triangles.*

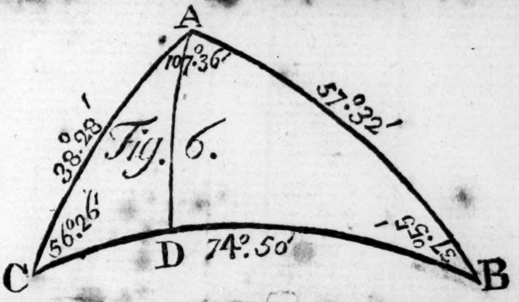


Fig. 6.

	Given	Sought	
	B	AC	S.C : S.AB : : S.B : S.AC. then
	C	A	$\text{Coſ.}\frac{AB-AC}{2} : \text{Coſ.}\frac{AB+AC}{2} :: \text{T.}\frac{C+B}{2} : \text{Cot.}\frac{A}{2}$ *
	AB	BC	$\text{Coſ.}\frac{C-B}{2} : \text{Coſ.}\frac{C+B}{2} :: \text{T.}\frac{AB+AC}{2} : \text{T.}\frac{BC}{2}$ ‡
I	B	AB	$\text{Coſ.}\frac{C+B}{2} : \text{Coſ.}\frac{C-B}{2} :: \text{T.}\frac{BC}{2} : \text{T.}\frac{AB+AC}{2}$ again
	C	AC	$\text{T.}\frac{C+B}{2} : \text{T.}\frac{C-B}{2} :: \text{T.}\frac{AB+AC}{2} : \text{T.}\frac{AB-AC}{2}$
	BC	A	S.AB : S.C : : S.BC : S.A.
	AB	A	$\text{Coſ.}\frac{BC+AB}{2} : \text{Coſ.}\frac{BC-AB}{2} :: \text{Cot.}\frac{B}{2} : \text{T.}\frac{A+C}{2}$ then
II	BC	C	$\text{T.}\frac{BC+AB}{2} : \text{T.}\frac{BC-AB}{2} :: \text{T.}\frac{A+C}{2} : \text{T.}\frac{A-C}{2}$
	B	AC	S.C : S.AB : : S.B : S.AC
	AB	B	S.AB : S.C : : S.AC : S.B. then
V	AC	A	are found as in *Caſe* I.
	C	BC	
	AB	A	$\text{T.}\frac{BC}{2} : \text{T.}\frac{AB+AC}{2} :: \text{T.}\frac{AB-AC}{2} : \text{T.}\frac{BD-DC}{2}$
	BC	B	
	AC	C	hence BD . DC are found, and the Angles by *Caſe* III. of right-angled Triangles . . . Or thus
V			$\text{S.AB}\times\text{S.AC} : \text{Rq} :: \text{S.X}\times\text{S.Z} : \text{Sq}\,\frac{A}{2}$
			and ſo for any other.
			N.B. $\frac{BC+AB+AC}{2}\left\{\begin{matrix}-AC=X\\-AB=Z\end{matrix}\right.$
	A	AB	Convert the Angles into Sides by *Cor.* to *Property* X. *page* 8. and proceed as in *Caſe* V.
VI	B	BC	
	C	AC	

* Lemma Th. II.   ‡ Lemma Th. IV.